2025

(May–June)

FYUP : 2nd Semester Examination

MATHEMATICS

(Major)

(Fundamental Mathematics—II)

(MTH–150)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **four** questions, selecting **one** from each Unit

UNIT—I

1. *(a)* If by rotation of the rectangular axes about the origin, the expression $ax^2+2hxy+by^2$ changes to $a'x'^2+b'y'^2+2h'x'y'$, prove that $a+b=a'+b'$ and $ab-h^2=a'b'-h'^2$. 5

(b) Find the transformed equation of the curve $(x+2y+4)(2x-y+5)=25$, when the two perpendicular lines $x+2y+4=0$ and $2x-y+5=0$ are taken as coordinate axes. 5

(c) Prove that the equation

$$2x^2+xy-6y^2-6x+23y-20=0$$

represents a pair of straight lines. Find the coordinates of their point of intersection and the angle between them. 5

(d) Find the diameter of the conic $8x^2-6xy+9y^2+3=0$ conjugate to the diameter $y-3x=0$. 3

2. (a) Find the centre of the conic

$$8x^2+9y^2-16xy-6x+4y+5=0$$ 2

(b) Reduce the equation

$$9x^2+24xy+16y^2-126x+82y-59=0$$

to standard form. 5

(c) If the two pairs of lines

$$x^2-2pxy-y^2=0$$

and $$x^2-2qxy-y^2=0$$

be such that each pair bisects the angles between the other pair, prove that $pq+1=0$. 5

(d) (i) Show that the diameters $y+3x=0$ and $4y-x=0$ are conjugate diameters of the ellipse $3x^2+4y^2=5$. 2

(ii) Prove that the parametric equations

$$x=\frac{a}{2}\left(\lambda+\frac{1}{\lambda}\right) \text{ and } y=\frac{b}{2}\left(\lambda-\frac{1}{\lambda}\right)$$

represent a hyperbola. 2

(iii) Find the equation of the tangent to the conic

$$x^2-8xy+2y^2-6x+10y-28=0$$

at the point (2, −3). 2

UNIT—II

3. (a) Find the equation of the plane passing through (1, −2, 1) and perpendicular to the line whose direction ratios are 2, 3, 5. 4

(b) Find the equation of the cone whose vertex is (α, β, γ) and whose base is the parabola $y^2=4ax$, $z=0$. 5

(c) Prove that the plane $ax + by + cz = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular generators if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$ where $abc \neq 0$. 5

(d) Show that the two circles

$$x^2 + y^2 + z^2 - y + 2z = 0$$
$$x - y + z - 2 = 0$$

and $$x^2 + y^2 + z^2 + x - 3y + z - 5 = 0$$
$$2x - y + 4z - 1 = 0$$

lie on the same sphere and find its radius. 5

4. (a) Obtain the equations of the circle lying in the sphere

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 3 = 0$$

and having its centre at (2, 3, –4). 5

(b) Find the equation of the plane through the line of intersection of the planes

$$2x - y + 3z + 5 = 0$$
$$x + 3y - 2z - 3 = 0$$

and perpendicular to the plane

$$3x - y - 2z + 7 = 0$$ 5

(c) Find the equation of the right circular cone whose vertex is (3, 2, 1), axis is the line

$$\frac{x-3}{4} = \frac{y-2}{1} = \frac{z-1}{3}$$

and semi-vertical angle is 30°. 4

(d) Find the equation of the sphere which touches the sphere

$$x^2 + y^2 + z^2 - 6x + 4y - 2z - 5 = 0$$

at the point (4, 1, –2) and passing through the origin. 5

UNIT—III

5. (a) State and prove Euler's theorem on homogeneous function of two variables. 1+4=5

(b) Show that

$$\underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x^4 + y^4}{x^2 + y^2} = 0$$ 4

(c) State Schwarz's theorem. If

$$f(x, y) = \log\left(\frac{x^2 + y^2}{xy}\right)$$

show that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. 1+4=5

(d) Determine whether the function f defined by

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is continuous at $(0, 0)$. 5

6. (a) Are the following functions homogeneous? If so, find their degree of homogeneity : 1+1=2

(i) $\frac{x}{y} + \tan^{-1}\left(\frac{y}{x}\right)$

(ii) $x^{-\frac{1}{3}}y^{\frac{4}{3}} - \tan\left(\frac{y}{x}\right)$

(b) If $u = 2\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$xu_x + yu_y + \cot\left(\frac{u}{2}\right) = 0.$$ 5

(c) If $u = \frac{x^2y^2}{x+y}$, apply Euler's theorem to find $xu_x + yu_y$ and hence deduce that

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 0$$ 3+3=6

(d) (i) If $z = f(ax + by)$, show that

$$b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = 0$$ 4

(ii) Find the domain of $z = e^{x-y}$, where $x, y \in \mathbb{R}$. 2

UNIT—IV

7. (a) If

$$\vec{a} = -3\hat{i} + 7\hat{j} + 5\hat{k}$$
$$\vec{b} = -5\hat{i} + 7\hat{j} - 3\hat{k}$$
$$\vec{c} = 7\hat{i} - 5\hat{j} - 3\hat{k}$$

calculate the volume of the parallelopiped having $\vec{a}$, $\vec{b}$, $\vec{c}$ as its adjacent sides with a common origin. Also find $\vec{a} \times (\vec{b} \times \vec{c})$ and $\vec{b} \times (\vec{c} \times \vec{a})$. 3+2=5

(b) Show that three non-zero vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar if and only if $\vec{a} \cdot \vec{b} \times \vec{c} = 0$. 3

(c) Prove that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ if and only if $(\vec{c} \times \vec{a}) \times \vec{b} = 0$. 3

(d) Prove that

$$(\vec{b} \times \vec{c}) \times (\vec{a} \times \vec{d}) + (\vec{c} \times \vec{a}) \times (\vec{b} \times \vec{d}) + (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = -2[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$$ 5

(e) Find $\frac{d}{dt}(\vec{a} \times \vec{b})$ if $\vec{a} = 2t\hat{i} + (t^2 + 1)\hat{j} + t\hat{k}$ and $\vec{b} = 2\sin t\hat{i} + t\hat{j} - \cos t\hat{k}$. 3

8. (a) If

$$\vec{r}_1 = \cos\theta\hat{i} + \sin\theta\hat{j} + \theta\hat{k}$$
$$\vec{r}_2 = \sin\theta\hat{i} - \cos\theta\hat{j} - 2\hat{k}$$
$$\vec{r}_3 = \hat{i} + 2\hat{j} - 3\hat{k}$$

find $\frac{d}{d\theta}(\vec{r}_1 \times (\vec{r}_2 \times \vec{r}_3))$ at $\theta = 0$. 5

(b) Find the unit tangent vector to the curve $x = 2t^2 + 4$, $y = 8t - 1$, $z = 6t^2 + 7t + 1$ at $t = 0$. Also, find the tangent line and normal plane to the curve at $t = 0$. 3+2=5

(c) Find the directional derivative of the function $f = xy + yz + zx$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$ at the point (1, 2, 0). 2

(d) Prove that $\vec{\nabla}\log|\vec{r}| = \frac{\vec{r}}{r^2}$, where $r = |\vec{r}|$. 3

(e) Evaluate

$$\frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2} \times \frac{d^3\vec{r}}{dt^3}$$

where $\vec{r} = (a\cos t,\ a\sin t,\ t\tan\alpha)$, a and α being constants. 4

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